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LETTER TO THE EDITOR

Heat kernel expansion for fermionic billiards in an external magnetic field

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Abstract. Using Seeley's heat kernel expansion, we compute the asymptotic density of states of the Dirac operator coupled to a magnetic field on a two-dimensional manifold with boundary ('fermionic billiard'). Local boundary conditions compatible with vector current conservation depend on a free parameter α. It is shown that the perimeter correction identically vanishes for $\alpha = 0$. In that case, the next-order constant term is found to be proportional to the Euler characteristic of the manifold. These results are independent of the external magnetic field and of the shape of the billiard, provided the boundary is sufficiently smooth. For the flat circular billiard, the constant term is found to be $-\frac{1}{2}$, in agreement with a numerical result of Berry and Mondragon.

In a recent work, Berry and Mondragon (1) have considered a Dirac Hamiltonian describing a massless spin-half particle moving on a finite domain in the plane. Hard-wall local boundary conditions obtained from the requirement that no current flows outwards from the billiard imply that the system does not possess time-reversal symmetry and is thus an interesting model in the context of quantum chaos. In order to extract the fluctuating part of the spectrum which in this case is shown to be distributed according to the statistics of the Gaussian unitary ensemble, the authors provide an explicit expression for the semiclassical integrated density of states. Surprisingly enough, the perimeter correction vanishes in this asymptotic expansion and numerical calculation suggests that the next-order constant term is $-\frac{1}{12}$. The purpose of this work is to extend this problem to the more general case of a two-dimensional curved manifold M bounded by a multiconnected smooth curve ∂M with some possible coupling to an external magnetic field. In contrast with Berry and Mondragon, who used the Balian–Bloch formalism, our approach is based on Seeley's heat kernel expansion [2].

Let $g_{\mu\nu}$ be the metric tensor of the Riemannian manifold M and $g = \det g_{\mu\nu}$. At each point of M, we introduce a zweibein e^a_μ such that

$$e^a_\mu e^a_\nu = g_{\mu\nu} \qquad g^{\mu\nu} e^a_\mu e^b_\nu = \delta^{ab}. \tag{1}$$

We consider Euclidean anti-Hermitic gamma matrices that satisfy

$$\{\gamma^a, \gamma^b\} = -2\delta^{ab} \qquad (\gamma^a)^\dagger = -\gamma^a. \tag{2}$$

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In two dimensions, one representation of these matrices is found in terms of the Pauli's σ matrices

$$\gamma^1 = \mathrm{i}\sigma^1 \qquad \gamma^2 = \mathrm{i}\sigma^2 \qquad \gamma^5 = \mathrm{i}\gamma^1\gamma^2 = \sigma^3. \tag{3}$$

The dynamics of the spin-$\frac{1}{2}$ particle is governed by the Euclidian Dirac equation on M:

$$\gamma^a e^a_\mu \, \mathrm{D}^\mu \psi(x) = E\psi(x). \tag{4}$$

The covariant derivative for the spinor $\mathrm{D}_\mu = \partial_\mu + \Gamma_\mu - \mathrm{i}qA_\mu$ takes into account the curvature of the manifold and the external magnetic field through the spin connection:

$$\Gamma_\mu = -\tfrac{1}{8}[\gamma^a, \gamma^b] e^{a,\nu} \nabla_\mu e^b_\nu \tag{5}$$

$$\Gamma_\mu = -\Gamma^\dagger_\mu \qquad \nabla_\mu e^b_\nu = \partial_\mu e^b_\nu - \Gamma^\rho_{\mu\nu} e^b_\rho \tag{6}$$

and through the vector potential A_μ to which the particle is coupled via its charge q, respectively.

Hermiticity of $\not{D} = e^{a\mu}\gamma^a \, \mathrm{D}_\mu$ is guaranteed provided the spinor $\psi(x)$ obeys the following class of boundary conditions:

$$n^\mu e^a_\mu \gamma^a \psi(x) = \gamma^5 \, \mathrm{e}^{\alpha\gamma^5} \psi(x) \qquad \text{on } \partial M \tag{7}$$

with α a free real parameter and n^μ the inward normal vector on ∂M. For $\alpha = 0$, one recovers the boundary condition considered by Berry and Mondragon [1]. One can then easily show that the vector current is conserved:

$$n^\mu(\psi^\dagger \gamma^a e^a_\mu \psi) = 0 \qquad \text{on } \partial M. \tag{8}$$

The axial symmetry, however, is explicitly broken, even for vanishing external field, in agreement with a theorem of Atiyah *et al* [3], which states that preservation of axial symmetry requires non-local boundary conditions. The same statement holds for time-reversal symmetry.

Introducing the projector

$$P_+ = \tfrac{1}{2}(1 + \gamma^5(\gamma n))$$

equation (7) can be rewritten in a more transparent way, namely

$$\mathrm{e}^{-(\alpha/2)\gamma^5} P_+ \, \mathrm{e}^{(\alpha/2)\gamma^5} \psi = \psi. \tag{9}$$

This demonstrates that through the chiral transformation $\psi' = \mathrm{e}^{(\alpha/2)\gamma^5}\psi$ one thus gets back the usual boundary conditions on ψ' with $\alpha = 0$. However, as we shall see later, the crucial point is that the spectrum is not invariant under this chiral transformation. (Similar features that arise in the context of the chiral bag models are responsible of the appearance of a non-vanishing baryon number [4].)

The knowledge of the asymptotic behaviour of the density of states $\rho(E)$ can be reached through the high-temperature expansion of the partition function

$$Z(\beta) = \mathrm{Tr}(\mathrm{e}^{-\beta \not{D}^2}) \tag{10}$$

which is related to $\rho(E)$ by a Tauberian theorem [5, 6].

The evaluation of the asymptotic expansion of $Z(\beta)$ as $\beta \to 0$ will be performed using Seeley's work [2]. Since we have to deal with the elliptic operator $\not{D}^2$ rather than $\not{D}$, a second boundary condition (in addition (7)) is needed, and we shall take

$$n^\mu e^a_\mu \gamma^a \, \not{D}\psi(x) = \gamma^5 \, \mathrm{e}^{\alpha\gamma^5} \, \not{D}\psi(x) \qquad \text{on } \partial M \tag{11}$$

which ensures that

$$\not{D}^2 = -g^{\mu\nu}(\nabla_\mu + \Gamma_\mu - \mathrm{i}qA_\mu)(\partial_\nu + \Gamma_\nu - \mathrm{i}qA_\nu) + X \tag{12}$$

with

$$X = \tfrac{1}{4}R - \tfrac{1}{4}\mathrm{i}qe^\mu_a e^\nu_b[\gamma^a, \gamma^b]F_{\mu\nu}$$

is Hermitian.

Now, let us briefly recall Seeley's heat kernel expansion [2], following the notation of [7]. Let D_B denote an elliptic operator of order ω acting on q-component functions, and M a ν-dimensional manifold bounded by a smooth boundary ∂M. The subscript B on D_B indicates a set of $[\omega q/2]$ boundary conditions. Since the expansion of $\mathrm{Tr}\{\mathrm{e}^{-\beta D_\mathrm{B}}\}$ involves only geometrical invariants of the manifold M [8], we can always choose to work on

$$M = \{(x, r) \mid x \in \mathbb{R}^{\nu-1}, r \in \mathbb{R}^+).$$

The first step consists in writing a Cauchy relation.

$$\mathrm{e}^{-\beta D_\mathrm{B}} = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \mathrm{d}\lambda \frac{\mathrm{e}^{-\beta\lambda}}{\lambda\mathbb{1} - D_\mathrm{B}} \tag{13}$$

where Γ is a contour enclosing all the eigenvalues of D_B. We thus now need to find the bounded propagator K^λ_B (bounded 'heat kernel' for the heat equation):

$$K^\lambda_\mathrm{B}(y, y') \equiv \left\langle y \left| \frac{1}{\lambda\mathbb{1} - D_\mathrm{B}} \right| y \right\rangle \tag{14}$$

which is a solution of the following set of equations:

$$(\lambda\mathbb{1} - D_\mathrm{B})_x K^\lambda_\mathrm{B}(x, y) = \delta(x - y) \qquad \forall x, y \in M \tag{15a}$$

$$B_j K^\lambda_\mathrm{B}(x, y) = 0 \qquad \text{on } \partial M \tag{15b}$$

$$j = 1, \ldots, [\omega q/2].$$

The trick is to write $K^\lambda_\mathrm{B} = K^\lambda - H^\lambda_\mathrm{B}$, where K^λ is the free propagator on $\mathbb{R}^\nu$

$$(\lambda\mathbb{1} - D_\mathrm{B})_y K^\lambda(y, y') = \delta(y - y') \qquad \forall y, y' \in \mathbb{R}^\nu \tag{16}$$

and H^λ_B satisfies

$$(\lambda\mathbb{1} - D_\mathrm{B})_y H^\lambda_\mathrm{B}(y, y') = 0 \qquad \forall y, y' \in M \tag{17}$$

$$B_j H^\lambda_\mathrm{B}(y, y') = B_j K^\lambda(y, y') \qquad \text{on } \partial M \tag{18}$$

$$H^\lambda_\mathrm{B}(x, r) \xrightarrow[r\to\infty]{} 0 \qquad x \in \mathbb{R}^{\nu-1}. \tag{19}$$

As a next step, one introduces the symbols of the above operator. Let $u(x, r)$ be a function on which D_B acts. Its Fourier transform is then defined by

$$u(x, r) = \frac{1}{(2\pi)^\nu}\int \mathrm{d}^{\nu-1}_\xi \int \mathrm{d}\tau\, \exp(\mathrm{i}\xi x + \mathrm{i}\tau r)\hat{u}(\xi, \tau). \tag{20}$$

For brevity, we set $r = x^\nu$ and $\tau = \xi^\nu$. Let $\alpha = (\alpha_1, \ldots, \alpha_\nu)$ be a multi-index with α_i non-negative integers and $\Sigma_i\, \alpha_i = |\alpha|$. Then, formally,

$$D = \sum_{|\alpha| \leqslant \omega} a_\alpha(x, r) D^\alpha_{x,r} \tag{21}$$

where $a_\alpha(x, r)$ is a set of $q \times q$ matrices and

$$D^\alpha_{x,r} = \prod_{i=1}^{\nu} \left(-\mathrm{i}\frac{\partial}{\partial x^i}\right)^{\alpha_i}. \tag{22}$$

The symbol of the operator D is defined as:

$$\sigma(D) = \sum_{|\alpha| \leq \omega} a_\alpha(x, r)(\xi, \tau)^\alpha \tag{23}$$

where

$$(\xi, \tau)^\alpha = \prod_{i=1}^{\nu} (\xi \mathrm{i})^{\alpha_i}. \tag{24}$$

Seeley's idea is to find an expansion for the symbols of the propagators K^λ and H^λ_{B}:

$$\sigma(K^\lambda) = \sum_{j=0}^{\infty} c_{-\omega-j}(x, r, \xi, \tau, \lambda) \tag{25}$$

$$\sigma(H^\lambda_{\mathrm{B}}) = \sum_{j=0}^{\infty} d_{-\omega-j}(x, r, \xi, \tau, \lambda) \tag{26}$$

where the $c_{-\omega-j}$ and the $d_{-\omega-j}$ have well defined homogeneity properties:

$$\begin{aligned} c_{-\omega-j}(x, r, \rho, \xi, \rho\tau, \rho^\omega\lambda) &= \rho^{-\omega-j} c_{-\omega-j}(x, r, \xi, \tau, \lambda) \\ d_{-\omega-j}(x, r/\rho, \rho, \xi, \rho\tau, \rho^\omega\lambda) &= \rho^{-\omega-j} d_{-\omega-j}(x, r, \xi, \tau, \lambda). \end{aligned} \tag{27}$$

The c and the d are obtained by solving algebraic and ordinary differential equations, respectively, which follow from equations (16)-(19). When these c and d are found, one can expand the bounded kernel K^λ_{B} in the Cauchy integral and use the change of variable (28) to find:

$$\lambda \to \lambda/\beta \tag{28}$$

$$\langle x, r|\mathrm{e}^{-\beta D_{\mathrm{B}}}|x, r\rangle \underset{\beta \to 0^+}{\approx} \sum_{j \geq 0} \beta^{(j-\nu)/\omega}[\varepsilon_j(x, r) + \eta_j(x, r)] \tag{29}$$

with

$$\varepsilon_j(x, r) = \frac{1}{(2\pi)^{\nu+1}} \int \mathrm{d}^{\nu-1}_\xi \int \mathrm{d}\tau \int \mathrm{d}s\, \mathrm{e}^{\mathrm{i}s} c_{-\omega-j}(x, r, \xi, \tau, -\mathrm{i}s) \tag{30}$$

$$\begin{aligned} \eta_j(x, r) = \frac{-1}{(2\pi)^{\nu+1}} &\sum_{k+n+1=j} \frac{(-1)^n \delta^{(n)}(r)}{n!} \\ &\times \int \mathrm{d}^{\nu-1}_\xi \int \mathrm{d}s\, \mathrm{e}^{\mathrm{i}s} \int \mathrm{d}r\, r^n \tilde{d}_{-\omega-k}(x, r, \xi, r, -\mathrm{i}s) \end{aligned} \tag{31}$$

and

$$\tilde{d}_{-\omega-k}(x, r, \xi, s, \lambda) = -\oint_{\Gamma_-} \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}s\tau} d_{-\omega-k}(x, r, \xi, \tau, \lambda). \tag{32}$$

Γ_- is a contour enclosing all the singularities of $d_{-\omega-k}$ in the lower complex-τ half plane. The ε terms are associated with the free kernel and the η give the boundary corrections. Seeley has shown that this expansion provides a good (asymptotic) approximation to the short-time (high-temperature) behaviour of $\mathrm{Tr}\{\mathrm{e}^{-\beta D_{\mathrm{B}}}\}$. (Note that the reason for considering symbols of operators rather than operators themselves is mainly that the Fourier transform takes partial differential equations into algebraic equations, which are easier to handle. For further details on Seeley's work, we refer the reader to the papers mentioned above.)

The terms associated with the free propagator are standard, since they only depend on geometrical invariants of the full manifold M. They can be found in [8, 9]. Here we give the first three terms we shall need:

$$\varepsilon_0=\frac{\sqrt{g}}{(4\pi)^{\nu/2}} \qquad \varepsilon_1=0 \qquad \varepsilon_2=\frac{\sqrt{g}}{(4\pi)^{\nu/2}}\left(\frac{R}{6}-X\right). \tag{33}$$

Our main goal is thus to compute the boundary terms for the fermionic billiard. We first need the d_{-2} term. Applying equation (17), one finds [7]

$$d_{-2}=c_1\,\mathrm{e}^{-\Lambda r}+c_2\,\mathrm{e}^{\Lambda r} \tag{34}$$

where we have set

$$\Lambda(x,\xi,\lambda)=[g_{22}(x,0)(g^{11}(x,0)\xi^2-\lambda)]^{1/2}. \tag{35}$$

Furthermore, equation (19) implies that the second term vanishes: $C_2=0$.

The determination of the constant C_1 goes through equation (18). It gives

$$\begin{aligned}C_1(x,\xi,\tau,\lambda)&=\frac{g_{22}\,\mathrm{e}^{-\alpha}}{2(\tau^2+\Lambda^2)(\sqrt{g^{22}}\,\Lambda\cosh\alpha-\sqrt{g^{11}}\,\xi\sinh\alpha)}\\&\quad\times\begin{pmatrix}(1-\mathrm{e}^{2\alpha})\sqrt{g^{11}}\xi+(\Lambda-\mathrm{i}\tau\,\mathrm{e}^{2\alpha})\sqrt{g^{22}} & \sqrt{g^{22}}\,\mathrm{e}^{\alpha}(\Lambda+\mathrm{i}\tau)\\ \sqrt{g^{22}}\,\mathrm{e}^{\alpha}(\Lambda+\mathrm{i}\tau) & (1-\mathrm{e}^{2\alpha})\sqrt{g^{11}}\xi+(\Lambda\,\mathrm{e}^{2\alpha}-\mathrm{i}\tau)\sqrt{g^{22}}\end{pmatrix}.\end{aligned} \tag{36}$$

Then, using equation (32), we obtain

$$\begin{aligned}&\tilde{d}_{-2}(x,r,\xi,s,\lambda)\\&\quad=\frac{\pi g_{22}\,\mathrm{e}^{-(r+s)\Lambda}}{\Lambda(\sqrt{g^{22}}\,\Lambda\cosh\alpha-\sqrt{g^{11}}\xi\sinh\alpha)}\\&\qquad\times\begin{pmatrix}-\sinh\alpha(\sqrt{g^{11}}\xi+\sqrt{g^{22}}\Lambda) & \sqrt{g^{22}}\Lambda\\ \sqrt{g^{22}}\Lambda & -\sinh\alpha(\sqrt{g^{11}}\xi\mp\sqrt{g^{22}}\Lambda)\end{pmatrix}.\end{aligned} \tag{37}$$

Equation (31) thus leads to

$$\begin{aligned}\eta_1(x,r)&=-\frac{1}{8}\sqrt{\frac{g_{11}(x,0)}{\pi}}\,\delta(r)\\&\quad\times[(1-\cosh\alpha)\mathbb{1}+\gamma^5(\sinh\alpha-\eta^{\mu}e^{a}_{\mu}\gamma^{a})]\end{aligned} \tag{38}$$

where we have used elementary Gaussian integrals and the formula

$$\int_{-\infty}^{\infty}\mathrm{d}s\frac{\mathrm{e}^{\mathrm{i}s}}{(x+\mathrm{i}s)^n}=\frac{2\pi\,\mathrm{e}^{-x}}{\Gamma(n)} \tag{39}$$

where $\Gamma(x)$ is the Euler gamma function.

The first boundary correction to the partition function is therefore

$$Z^{1\mathrm{B}}(\beta)\equiv\int_M\mathrm{d}x\,\mathrm{d}r\,\mathrm{Tr}\{\eta_1(x,r)\}=\frac{\mathscr{L}(\cosh\alpha-1)}{4\sqrt{\pi\beta}} \tag{40}$$

where $\mathscr{L}$ stands for the perimeter of ∂M. As Berry and Mondragon pointed out, this correction vanishes with the boundary parameter $\alpha = 0$. Note that the boundary contribution (40) is independent both of the shape of the boundary (only the smoothness of each connected component of ∂M is required) and of the external magnetic field.

In the special case where $\alpha = 0$, we thus need the second boundary correction to the partition function in order to obtain the deviation from the free case. This is not necessary in the general case because this correction will give a constant term independent of the energy in the asymptotic expansion of the density of states, which is then negligible compared with the first term of (40), of order $E^{1/2}$.

Following the same procedure as above, we obtain, after some calculations, the expression of the second boundary correction term η_2:

$$\eta_2(x, r) = -\frac{\delta(r)}{8\pi}\left\{-\mathrm{i}\sqrt{\gamma(x)}\,\gamma^a e^{a\mu} g_{\mu\nu} t^\rho(\partial_\rho n^\nu)\right.$$
$$\left.+\frac{\sqrt{\gamma(x)}}{3}K(x)\left[\mathbb{1}+\frac{5\mathrm{i}}{2}t^\mu e^a_\mu \gamma^a\right]\right\} - \mathrm{i}\frac{\delta'(r)}{8\pi}\sqrt{\gamma(x)}\,t^\mu e^a_\mu \gamma^a \tag{41}$$

where t_μ and n_μ are the tangent and inward normal unit vectors on ∂M respectively. $K(x)$ is the second fundamental form on ∂M and $\gamma(x)$ is the determinant of the induced metric on ∂M [10].

Thus, the second boundary correction to the partition function is

$$Z_{2\mathrm{B}}(\beta) \equiv \int_M \mathrm{d}x\,\mathrm{d}r\,\mathrm{Tr}\,\eta_2(x, r) = -\frac{1}{12\pi}\int_{\partial M}\mathrm{d}s\,K(s). \tag{42}$$

Putting all terms together, we obtain the final form of the partition function:

$$Z(\beta) = \frac{t}{2\pi\beta} - \frac{1}{12\pi}\left(\frac{1}{2}\int_M \mathrm{d}^2x\sqrt{g}\,R + \int_{\partial M}\mathrm{d}s\,K(s)\right) + \mathrm{O}(\sqrt{\beta}). \tag{43}$$

Using the Gauss-Bonnet theorem:

$$\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\,R + \int_{\partial M}\mathrm{d}s\,K(s) = 2\pi\chi(M) \tag{44}$$

where χ is the Euler characteristic of the manifold M, we find:

$$Z(\beta) = 2\left(\frac{t}{4\pi\beta} - \frac{\chi(M)}{12}\right) + \mathrm{O}(\sqrt{\beta}). \tag{45}$$

Then from equations (40) and (45) and a Tauberian theorem [5, 6], we are led to the asymptotic form of the integrated level density $N(E)$:

$$\alpha \neq 0: \qquad N(E) = 2\left(\frac{t}{4\pi}E + \frac{\ell(\cosh\alpha - 1)\sqrt{E}}{4\pi}\right) + \mathrm{O}(1) \tag{46}$$

$$\alpha = 0: \qquad N(E) = 2\left(\frac{t}{4\pi}E - \frac{\chi(M)}{12}\right) + \mathrm{O}\left(\frac{1}{\sqrt{E}}\right). \tag{47}$$

Now, $\chi = 2 - 2h - b$, where h is the number of handles of the manifold M and b is the number of connected components of the boundary ∂M. Thus, for the flat circular and 'Africa-shaped' billiards, for which $\chi = 1$, the integrated density of states is

$$N(E) = 2\left(\frac{t}{4\pi}E - \frac{1}{12}\right) + \mathrm{O}(E^{-1/2}) \tag{48}$$

in agreement with the numerical result of Berry and Mondragon, up to the factor 2 which just reflects the fact that we consider the spectrum of $\not{D}^2$ rather than $\not{D}$. Our result, however, is more general and applies to 2D manifolds with an arbitrary number of (smooth) boundaries and handles.

Up to the order considered here, the asymptotic expansion of $Z(\beta)$ is independent of the external field. Dependences with respect to A_μ appear, e.g., in the bulk contribution of order β (or E^{-1}), ε_4, which has been computed by Gilkey [8], and quite certainly also in the higher-order boundary contribution which, to our knowledge, have not been computed so far for the fermionic billiard. One can, however, conclude from the present analysis that the leading contributions to $N(E)$ depend only on the topology of the billiard and are insensitive to the presence of an external magnetic field.

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